PMM U.S.S.R.,Vol.46,pp.185-189
0021-8928/83/2 0185 $7.50/0

UDC 532.135:532.5.011

SLOW MOTION OF A SPHERE THROUGH AN ANISOTROPIC, VISCOELASTIC FLUID*

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Slow motion of a rigid sphere through an incompressible, linear anisotropic viscoelastic fluid of general type is studied. Expressions for the force and moment acting from the fluid towards the sphere are obtained. The results obtained can be used in constructing a theory of viscoelasticity of concentrated polymer solutions and melts where the mobility of the polymer chains is strongly anisotropic.

1. Using the representations of the statistical nonequilibrium mechanics the authors of /1/ have shown that the liner viscoelastic behavior of a large class of the incompressible fluids can be described by the following equations:

$$\sigma_{ij}(\mathbf{r},t) = -p\delta_{ij} + \int_{-\infty}^{t} ds\,\eta_{ijkl}(t-s)\,\gamma_{kl}(\mathbf{r},s) \tag{1.1}$$

Here p is the pressure and γ_{kl} is the deformation rate tensor. From the phenomenological point of view the equations (1.1) define a linear anisotropic viscoelastic fluid the properties of which are direction-dependent and described by the fourth rank tensor $\eta_{ijkl}(t)$ of the memory functions. The tensor of the memory functions has a number of properties determined by its symmetry conditions. By virtue of the symmetry of the stress and deformation rate tensors the tensor η_{ijkl} is symmetrical with respect to the indices i, j, kl and its invariance under the time inversion leads to the condition of symmetry in the form of a generalized Onsager theorem. We have

$$\eta_{ijkl}(t) = \eta_{jikl}(t) = \eta_{ijlk}(t), \qquad \eta_{ijkl}(t) = \eta_{klij}(t) \tag{1.2}$$

and this reduces the number of the independent memory functions to 21.

The functions can be further reduced using the symmetry group characterizing the form of the anisotropic medium. The memory functions tensor is invariant with respect to this symmetry group, and can therefore be represented in the form of a sum of a finite number of tensors with scalar coefficients. The general form of such tensors of up to the fourth rank inclusive is given in /2/ for any symmetry group.

Amongst the memory functions only U functions are independent, and these can always be written in the form /3/

$$\eta_{ijkl}(t) = \sum_{\alpha=1}^{V} c_{ijkl}^{\alpha}\eta_{\alpha}(t), \qquad 1 \leqslant V \leqslant U$$

Here c_{ijkl}^{α} are tensor constants and $\eta_{\alpha}(t)$ are certain scalar functions which can be written, in the case of fluids with discrete relaxation spectra, in the form of a finite sum of the indices with the relaxation times $\tau_{\alpha\gamma}$:

$$\eta_{\alpha}(t) = \sum_{\gamma} \frac{\eta_{\alpha\gamma}}{\tau_{\alpha\gamma}} \exp\left(-\frac{t}{\tau_{\alpha\gamma}}\right) \tag{1.3}$$

In the case of isotropic incompressible viscoelastic fluids the properties of which are invariant under all transformations of the orthogonal group, the tensor of the memory functions has the form

$$\eta_{ijkl}(t) = \eta(t)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}); \qquad \eta(t) = \sum_{\alpha=1}^{N} \frac{\eta_{\alpha}}{\tau_{\alpha}} \exp\left(-\frac{t}{\tau_{\alpha}}\right) \tag{1.4}$$

where η_{α} and τ_{α} are the relaxation viscosities and relaxation times respectively.

For a linear viscous anisotropic fluid the relaxation times $\tau_{\alpha\gamma} \to 0$ in (1.3) and the memory functions tensor has the form

$$\eta_{ijkl}(t) = 2\eta_{ijkl}\,\delta(t)$$

*Prikl.Matem.Mekhan.,46,No.2,pp.248-253,1982

In this case the properties of the medium are defined by a constant fourth rank tensor η_{ijkl} representing a viscosity coefficents tensor /4,5/, and $\delta(t)$ is the delta function.

Let us consider a motion of a sphere of radius a through a linear, anisotropic viscoelastic fluid of general type (1.1), at rest at infinity. We denote by $\mathbf{u}(t)$ the translational velocity of the sphere, and by $\boldsymbol{\omega}(t)$ its instantaneous rate of rotation, and introduce a coordinate system attached to the center of the sphere. If the translational and rotational Reynold's numbers are both small, which happens in many cases of practical importance /6/, then a quasistationary form of the system of equations of slow motion may be used (the time t appears as a parameter, and $\mathbf{r}$ is the local radius-vector measured relative to the center of the sphere)

$$v_{i,i} = 0, \quad \sigma_{ij,j} = 0, \quad \sigma_{ij}(\mathbf{r}, t) = -p\delta_{ij} + \int_{-\infty}^{t} ds\eta_{ijkl}(t-s)\gamma_{kl}(\mathbf{r}, s) \tag{1.5}$$

The velocity of perturbed motion of the fluid satisfies the condition of adhesion and of vanishing at infinity

$$v_i = u_i + \varepsilon_{ijk}\omega_j x_k,\ r = a;\ v_i \to 0,\ r \to \infty \tag{1.6}$$

Let us determine, with help of the problem (1.5), (1.6), the force and moment acting from the direction of the linear, anisotropic viscoelastic fluid, on the sphere moving through the fluid. The solution of the linear viscoelastic problem can be reduced by means of a Fourier transformation to the solution of the corresponding viscous problem. Indeed, the Fourier transformation

$$a(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} a(t)\, dt$$

by removing the time variable reduces the system (1.5) to the form

$$v_{i,i}(\mathbf{r}, \omega) = 0,\ \sigma_{ij,j}(\mathbf{r}, \omega) = 0, \quad \sigma_{ij}(\mathbf{r}, \omega) = -p\delta_{ij} + \eta_{ijkl}[\omega]\gamma_{kl}(\omega) \tag{1.7}$$

The above equations differ from the corresponding equations of the viscous problem only by the fact that the tensor η_{ijkl} is replaced by $\eta_{ijkl}[\omega]$, with ω appearing as a parameter.

In the linear formulation the velocity and pressure of the perturbed fluid can be written in the form

$$v_i = v_i' + v_i'', \quad p = p' + p''$$

The translational $(\mathbf{v}', p')$ and rotational $(\mathbf{v}'', p'')$ fields satisfy the initial equations of motion and the corresponding boundary conditions

$$\begin{aligned} v_i' &= u_i,\ r = a;\ v_i' \to 0,\ r \to \infty \\ v_i'' &= \varepsilon_{ijk}\omega_j x_k,\ r = a;\ v_i'' \to 0,\ r \to \infty \end{aligned} \tag{1.8}$$

The total reaction force and moment acting on the sphere are also determined, generally speaking, in the form of a sum

$$\begin{aligned} F_i &= \int_S ds_j\sigma_{ji}' + \int_S ds_j\sigma_{ji}'' \\ M_i &= \varepsilon_{ij\beta}\int_S ds_\alpha x_j\sigma_{\alpha\beta}' + \varepsilon_{ij\beta}\int_S ds_\alpha x_j\sigma_{\alpha\beta}'' \end{aligned} \tag{1.9}$$

Here σ_{ij}' and σ_{ij}'' are the stress tensors connected with the translational and rotational motion respectively, and $\varepsilon_{ij\beta}$ is an alternating tensor. We note that the total moment $\mathbf{M}$ depends, in contrast to the total force $\mathbf{F}$, on the choice of the point from which the local radius vector is measured.

2. In the present case the translational field $(\mathbf{v}', p')$ depends on the properties of the fluid determined by the fourth rank tensor η_{ijkl}, and also on the vector $\mathbf{u}$ characterizing the translational motion of the sphere. Since the translational motion equations are linear, we can follow the example of the case of an arbitrary body in an isotropic medium /7/ and introduce a characteristic tensor of the translational velocity field V_{ij}' and the associated characteristic vector of the pressure field P_i', which are independent of those parameters, as follows:

$$v_i' = V_{is}', \quad p'\delta_{ij} = \frac{1}{2}\eta_{ijlm}\delta_{lm}P_s'u_s \tag{2.1}$$

The incompressibility condition yields

$$\eta_{ijll} = \mu\delta_{ij}$$

and the above equations must be regarded as additional constraints imposed on the components of the viscosity tensor.

The dependence of V_{ij}' and P_i' on the radius-vector of the point is determined by the geometrical properties of the sphere surface only. From (2.1), (1.7) it follows that in this case a characteristic tensor of the stress field Π_{lms}' also exists, connected with the translational motion and such that

$$\sigma_{ij}' = \frac{1}{2}\eta_{ijlm}\Pi_{lms}'u_s\,,\quad \Pi_{lms}' = -\delta_{lm}P_s' + \partial V_{ls}'/\partial x_m + \partial V_{ms}'/\partial x_l \tag{2.2}$$

(Π_{lms}' depends, just as V_{ij}' and P_j', only on the geometry of the body).

Let us obtain the equations which satisfy the characteristic fields of higher order given here. Substituting (2.1) and (2.2) into the initial equations of translational motion and the boundary conditions, we have

$$\frac{\partial}{\partial x_i}V_{il}' = 0,\quad \frac{\partial^2}{\partial x_j\,\partial x_j}V_{ls}' = \frac{\partial}{\partial x_l}P_s',\quad V_{ij}' = \delta_{ij},\ r = a;\ V_{ij}' \to 0,\ r\to\infty \tag{2.3}$$

In deriving the equations and ocnditions (2.3) we have utilized the equations connecting the vectors and tensors, remembering that u_l and η_{ijkl} are arbitrary. The fact that the equations obtained here are independent of the medium properties and the sphere motion velocity, proves the existence of the characteristic fields introduced here.

For a sphere of radius a the characteristic field of translational motion has the form /7/

$$V_{ij}' = \frac{3}{4}\frac{a}{r}(n_in_j + \delta_{ij}) - \frac{3}{4}\left(\frac{a}{r}\right)^3\left(n_in_j - \frac{1}{3}\delta_{ij}\right),\quad P_i' = \frac{3}{2}\frac{a}{r^2}n_i,\quad n_i = \frac{x_i}{r} \tag{2.4}$$

From (2.2) and (2.4) we obtain the characteristic stress tensor connected with the translational motion of the sphere

$$\Pi_{lms}' = -\frac{3}{2}\frac{a^3}{r^4}(\delta_{lm}n_s + \delta_{ls}n_m + \delta_{sm}n_l) - \frac{3}{2}\frac{a}{r^4}(3r^2 - 5a^2)\,n_ln_mn_s$$

In accordance with (1.9) and (2.2.), the resistance of the sphere during its translational motion can be expressed in terms of Π_{lms}' as follows:

$$F_i = \frac{1}{2}\eta_{iklm}\int_S ds_k\Pi_{lms}'u_s \tag{2.5}$$

Let us introduce the characteristic tensor of hydrodynamic resistance of the translational motion

$$K_{klms} = -\frac{1}{2}\int_S ds_k\Pi_{lms}' \tag{2.6}$$

depending, just as Π_{lms}', on the dimension and form of the body in question. Substituting the expression for Π_{lms}' into (2.6) and integrating over the surface of the sphere of radius a, and using the identities

$$\int_\Omega n_in_j\,d\Omega = \frac{4\pi}{3}\delta_{ij},\quad \int_\Omega n_in_jn_kn_l\,d\Omega = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{2.7}$$

where Ω is the surface of a unit radius sphere, we obtain

$$K_{klms} = {}^3/_5\pi a\,(\delta_{kl}\delta_{ms} + \delta_{sk}\delta_{ml} + \delta_{sl}\delta_{mk}) \tag{2.8}$$

From (2.5), (2.6) and (2.8) we find the force of resistance of a sphere during its translational motion through an anisotropic viscous fluid

$$F_i = -\zeta_{is}u_s,\quad \zeta_{is} = {}^3/_5\pi a\,(\eta_{isll} + \eta_{ilsl} + \eta_{ills})$$

The friction coefficients tensor ζ_{is} is symmetric, since the viscosity coefficients tensor is symmetric.

In the same manner we can show that the part of the force of reaction due to the rotation of the sphere in an anisotropic fluid, is equal to zero. It follows therefore, just as in the case of an isotropic environment, that the total hydrodynamic resistance of the sphere is independent of its rotation and reduces to the force of resistance to the translational motion.

We use the principle of correspondence of the viscous and viscoelastic linear problems to find the Fourier component of the force sphere resistance in the linear, viscoelastic anisotropic fluid

$$F_i(\omega) = -\zeta_{is}[\omega]\, u_s(\omega), \quad \zeta_{is}[\omega] = \frac{3}{5}\pi a\,(\eta_{isll}[\omega] + 2\eta_{ills}[\omega]) \tag{2.9}$$

The tensor of complex friction coefficients $\zeta_{is}[\omega]$ is written with the symmetry conditions (1.2) taken into account. The force itself is found, using the inverse Fourier transformation (2.9), in the form of a linear functional of the sphere velocity

$$F_i(t) = -\int_{-\infty}^{t} \zeta_{il}(t-s)\, u_l(s)\, ds$$

and in this case every component of the reaction tensor is a linear integral operator.

3. We analyze the rotational motion in the same manner. Since the equations of rotational motion are linear, the flow field $(\mathbf{v}'', p'')$ and the corresponding stress tensor σ_{ij}'' can be represented in the form

$$v_i'' = V_{is}''\omega_s, \quad p''\delta_{ij} = \frac{1}{2}\eta_{ijlm}\delta_{lm}P_s''\omega_s, \quad \sigma_{ij}'' = \frac{1}{2}\eta_{ijlm}\Pi_{lms}''\omega_s, \quad \Pi_{lms}'' = -\delta_{lm}P_s'' + \partial V_{ls}''/\partial x_m + \partial V_{ms}''/\partial x_l \tag{3.1}$$

The characteristic vector P_s'' and tensor V_{is}'' of the rotational field depend, at the given point, only on the radius of the sphere, and are given by the following equations and boundary conditions:

$$\frac{\partial}{\partial x_i}V_{is}'' = 0, \quad \frac{\partial^2}{\partial x_j\,\partial x_j}V_{ls}'' = \frac{\partial}{\partial x_l}P_s'', \quad V_{ij}'' = \varepsilon_{ijk}x_k, \quad r = a; \quad V_{ij}'' \to 0, \quad r \to \infty \tag{3.2}$$

For a sphere of radius a, the characteristic velocity and pressure fields satisfying (3.2) have the form /7/

$$V_{ij}'' = \varepsilon_{ijk}x_k\,(a/r)^3, \quad P_s'' = 0 \tag{3.3}$$

From (3.1) and (3.3) we obtain the characteristic stress tensor Π_{lms}'' connected with the rotational motion of the sphere

$$\Pi_{lms}'' = -3\,(a/r)^3\,(\varepsilon_{lsn}n_n n_m + \varepsilon_{msn}n_n n_l) \tag{3.4}$$

The moment of the reaction forces acting on the sphere rotating about its center is determined in accordance with (3.1) and (2.9) in terms of the characteristic stress tensor, in the following manner:

$$M_i'' = \frac{1}{2}\eta_{\alpha\beta lm}\int_S ds\,\varepsilon_{ij\beta}x_j n_\alpha \Pi_{lms}''\omega_s \tag{3.5}$$

Let us introduce the rotational tensor

$$\Omega_{i\alpha\beta lms} = -\frac{1}{2}\int_S ds\,\varepsilon_{ij\beta}x_j n_\alpha \Pi_{lms}'' \tag{3.6}$$

which, unlike the translational tensor $\mathbf{K}$ introduced earlier, depends on the choice of the coordinate origin. Substituting (3.4) into (3.6) and integrating we find, using the identities (2.7), that the rotational tensor at the sphere center is equal to

$$\Omega_{i\alpha\beta lms} = {}^2/_5\pi a^3\varepsilon_{ij\beta}\,(\varepsilon_{lsn}\Delta_{j\alpha nm} + \varepsilon_{msn}\Delta_{j\alpha nl})$$
$$\Delta_{ijkl} = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}$$

In this manner we obtain the following expression for the moment of the forces acting on the sphere rotating about its diameter in a viscous anisotropic fluid:

$$M_i = -\zeta_{is}^r\omega_s, \quad \zeta_{is}^r = {}^4/_5\pi a^3[(2\eta_{lmlm} - \eta_{llmm})\delta_{is} + 2\eta_{isll} - 3\eta_{ills}] \tag{3.7}$$

The tensor of rotational resistance coefficients ζ_{is}^r is symmetric by virtue of the symmetry of the viscosity coefficients. This means that three principal forces opposing the rotation exist. It can be shown that the moment acting on the sphere during its translational motion through an anisotropic fluid is equal to zero. It follows therefore that the total moment acting on the sphere can be reduced to the moment given by the formulas (3.7). Consequently the translational and rotational motions of a sphere in an anisotropic fluid are independent of each other just as in the case of an isotropic medium. Further, following the principle of correspondence of the viscous and viscoelastic problems we repalce in (3.7) η_{ijkl} by $\eta_{ijkl}[\omega]$ and obtain the following expression for the Fourier component of the moment of the reaction forces acting on the sphere in a linear, anisotropic viscoelastic fluid

$$M_i(\omega) = -\zeta_{is}^r[\omega]\,\omega_s(\omega), \quad \zeta_{is}^r[\omega] = {}^4/_5\pi a^3[(2\eta_{lmlm}[\omega] - \eta_{llmm}[\omega])\,\delta_{is} + 2\eta_{isll}[\omega] - 3\eta_{ills}[\omega]] \tag{3.8}$$

Inverse Fourier transform applied to (3.8) yields the formula for the moment of the resistance forces

$$M_i(t) = -\int_{-\infty}^{t} \zeta_{il}^r(t-s)\,\omega_l(s)\,ds \tag{3.9}$$

In the particular case of a sphere moving through an isotropic linear viscoelastic fluid (1.4), the results obtained coincide with the known results of /8,9/. The general formulas given in the paper for the resistance (2.5) and (2.6) and moment (3.5) and (3.6) forces hold also in the case of an arbitrary body moving slowly with velocity varying slowly with time, through a linear anisotropic viscoelastic medium.

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Translated by L.K.